A Gauge-Invariant Regularization of the

Weyl Determinant Using Wavelets [*]

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ABSTRACT

In line with a previous paper, a gauge-invariant regularization is developed for the Weyl determinant of a Euclidean gauged chiral fermion. We restrict ourselves to gauge configurations with the A field going to zero at infinity in Euclidean space; and thus restrict gauge transformations to those with U the identity at infinity. For each finite cutoff one gets a strictly gauge-invariant expression for the Weyl determinant. Full Euclidean invariance is only to be sought in the limit of removing the cutoff. We expect the limit to be Euclidean invariant, but this has not yet been proved. One need not enforce the no-anomaly condition on the representation of the gauge group! We leave to future research relating the present results to conventional physics wisdom.

INTRODUCTION

Following the development of a gauge-invariant regularization for a gauged scalar boson, and a gauged dirac fermion in [1], we continue in a similar vein with the treatment of a chiral gauged fermion by study of the Weyl determinant. The most surprising aspect of the study is that we are not forced to impose a no-anomaly condition on the group representation (at least for the results obtained so far).

The traditional treatment of the Weyl determinant is by Leutwyler in [2]. Further research will be required to relate our results to the usual textbook statements. But it seems certain that at the very least new insights into the role of anomalies will be uncovered.

AN ASIDE

It perhaps should be noted that there are "cheap" ways to obtain gauge-invariant regularizations. For example, one could define the determinant as to be calculated in the radial axial gauge about the origin. This is tautologically a gauge invariant definition. (It is blatantly not Euclidean invariant.) Then one can employ momentum cutoffs. But this procedure is very far from having the *localization* property described in Remark 3) at the end of the paper. There is no reason to expect renormalization to be implementable using *local* counterterms.

BASIC FACTS ABOUT THE WEYL OPERATOR

In four dimensional Euclidean space we study the Weyl operators

$$W_\pm = \sum_{j=1}^{3} \sigma_j(\partial_j + A_j) \pm i(\partial_0 + A_0) \tag{1}$$

with σ_j the Pauli matrices. The A_μ are anti-hermitian, that is iA_μ is hermitian. With $*$

representing conjugate-transpose one has

$$W_\pm^* = -W_\mp .$$ (2)

We first briefly consider the special case of a real representation of the gauge group. That is, the representation is equivalent to its conjugate. For simplicity we write the following two equations, (3) and (4), in the special case when A_μ is real. Using c to denote conjugation we then get

$$\sigma_2 \, W_\pm^c \, \sigma_2 = -W_\pm .$$ (3)

From (2) and (3) we see that for a real representation one has

$$\sigma_2 \, W_\pm^T \, \sigma_2 = W_\mp$$ (4)

with T indicating transpose. For a real representation thus one has

$$\det(W_+) = \det(W_-)$$ (5)

and one can obtain the Weyl determinants from the square root of the Dirac operator determinant. For non-real representations only the magnitude of the Weyl determinant may be deduced from the Dirac determinant. Explicitly

$$i \, D \; = \; i \begin{pmatrix} 0 & W_+ \\ W_- & 0 \end{pmatrix}$$ (6)

in a suitable representation of the γ matrices. And it follows

$$\det(i \, D) = det(W_+) \det(W_-)$$ (7)

from which the relationship between the Weyl and Dirac determinants. In what follows we study W_+ abbreviated as W.

THE WAVELET BASES

We follow the notation of Chapter 2 of [1]. We let $\psi_\alpha(x)$ be the wavelet basis constructed by Y. Meyer, orthonormal in the usual inner product.

$$\langle \psi_\alpha, \psi_\beta \rangle = \int d^4x \, \psi_\alpha(x) \, \psi_\beta(x) = \delta_{\alpha,\beta}.$$ (8)

The ψ_α carry suppressed 2-spinor and group indices. We let ψ_α^1, (a non-orthonormal basis) be defined by

$$\psi_\alpha^1 = \frac{1}{W_0}\,\psi_\alpha \tag{9}$$

with W_0 the operator

$$W_0 = \sum_{j=1}^3 \sigma_j \partial_j + i\partial_0 \ . \tag{10}$$

The Weyl determinant is then first viewed as the determinant of the infinite discrete matrix $M_{\alpha,\beta}$:

$$\det(W) \ = \ \det(M_{\alpha,\beta}) \tag{11}$$

$$M_{\alpha,\beta} \ = \ \left\langle \psi_\alpha,\ W\,\psi_\beta^1 \right\rangle . \tag{12}$$

THE CHANGE OF BASIS

We follow [1] and now change the bases used in (12), defining

$$\phi_\alpha(x) = u(x,\gamma_\alpha)\psi_\alpha(x) \tag{13}$$

and

$$\phi_\alpha^1(x) = u(x,\gamma_\alpha)\psi_\alpha^1(x) \tag{14}$$

These definitions mimic equation (4.4) in [1] using (3.8) of [1] to define $u(x,\gamma_\alpha)$, and recalling from Chapter 2 of [1] that γ_α is the "center" of ψ_α. Each wavelet has been put in the radial axial gauge about its center by the gauge transformations in (13) and (14). It must be noted that the gauge transformation is different from wavelet to wavelet, since different wavelets (may) have different centers. Changing bases in (12) we get now for the Weyl determinant:

$$\det(W) = \det(M_{\alpha,\beta}) \ = \ \det(N_{\alpha,\beta}) / \Big(\det(A_{\alpha,\beta}) \det(B_{\alpha,\beta}) \Big) \tag{15}$$

with

$$N_{\alpha,\beta} \ = \ \left\langle \phi_\alpha, W\phi_\beta^1 \right\rangle \tag{16}$$

$$A_{\alpha,\beta} \ = \ \left\langle \phi_\alpha, \psi_\beta \right\rangle \tag{17}$$

$$B_{\alpha,\beta} \ = \ \left\langle W_0^*\psi_\alpha, \phi_\beta^1 \right\rangle . \tag{18}$$

We now seek gauge-invariant representations for the numerator and denominator in (15), that is gauge-invariant cutoffs for the corresponding infinite determinants.

THE NUMERATOR DETERMINANT

If we restrict the indices in $N_{\alpha,\beta}$ to any finite set, the truncated matrix $N_{\alpha,\beta}^{TR}$ has a gauge-invariant determinant. This is as in the study of $n_{\alpha,\beta}$ in [1], in equations (4.12)-(4.23) therein. Actually with our conditions on the potential $A_\mu(x)$ we need only an ultraviolet cutoff, restricting ourself to wavelets with length scale, ℓ, greater than some cutoff, ℓ_0.

THE DENOMINATOR DETERMINANTS

We proceed to study the denominator in (15). We first define

$$C_{\alpha,\beta} = \left\langle \psi_\alpha, \phi_\beta \right\rangle \tag{19}$$

$$D_{\alpha,\beta} = \left\langle \phi_\alpha, \phi_\beta \right\rangle. \tag{20}$$

We write the denominator determinants

$$\det(A)\det(B) = L \cdot R \tag{21}$$

with

$$L = \det(A)\det(C) = \det(D) \tag{22}$$

and

$$R = \det(B)/\det(C) = \frac{\det(\left\langle W_0^* \psi_\alpha , \phi_\beta^1 \right\rangle)}{\det(\langle \psi_\alpha , \phi_\beta \rangle)}. \tag{23}$$

L is simple to deal with; for any finite truncation of D, $\det(D)$ is gauge-invariant similar to $B(0,0)$ in (4.28) of [1]. The treatment is as in (4.38)-(4.43) of [1]. Again we will only need an ultraviolet cutoff.

The study of R is more difficult and more interesting . One would like to imitate the development in [1] beginning with equation (4.24). There one interpolated between

$(-\Delta + m^2)$ and 1 using operators $(-\Delta + m^2)^s$, $0 \leq s \leq 1$. It is the lack of a similar suitable interpolation between W_0 and 1 that makes the chiral situation less elegant. But we begin with a preliminary interpolation.

AN INTERPOLATION STEP

We write R as a quotient

$$R = \frac{X}{Y} \tag{24}$$

with

$$X = \det\left(\langle W_0^* \psi_\alpha, \phi_\beta^1 \rangle\right) / \det\left(\langle \psi_\alpha^{(-1)}, \phi_\beta^{(1)} \rangle\right) \tag{25}$$

$$Y = \det\left(\langle \psi_\alpha, \phi_\beta \rangle\right) / \det\left(\langle \psi_\alpha^{(-1)}, \phi_\beta^{(1)} \rangle\right) \tag{26}$$

having introduced

$$\psi_\alpha^{(s)} = \frac{1}{(-\Delta)^{s/2}} \, \psi_\alpha \tag{27}$$

$$\phi_\alpha^{(s)} = u(x, \gamma_\alpha) \, \psi_\alpha^{(s)} \, . \tag{28}$$

These interpolating functions enable Y to be treated as in [1], and yield a manifestly gauge-invariant development for Y, with ultraviolet cutoffs yielding a gauge-invariant regularization. We have isolated all the new features and difficulties into X.

THE NEW FEATURE

Again we write X as a ratio

$$X = \frac{E}{F} \tag{29}$$

with

$$F = \det\left(\langle \phi_\alpha^{(-1)}, \psi_\beta^{(1)} \rangle\right) \det\left(\langle \psi_\alpha^{(-1)}, \phi_\beta^{(1)} \rangle\right) \tag{30}$$

$$= \det\left(\left\langle \phi_\alpha^{(-1)}, \phi_\beta^{(1)} \right\rangle\right) \tag{31}$$

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and

$$E \;=\; \det\left(\langle \phi_\alpha^{(-1)}, \psi_\beta^{(1)}\rangle\right) \det\left(\langle W_0^* \psi_\alpha, \phi_\beta^1\rangle\right) \tag{32}$$

$$\sim \;\; \det\left(\langle \phi_\alpha^{(-1)}, \phi_\beta^1\rangle\right) \tag{33}$$

F is as simple to treat as $\det(D)$ before, immediately of the form developed in [1]. The proportionality in (33) indicates a numerical factor independent of the gauge field.

Collecting the expression we have obtained so far for the Weyl determinant

$$\det(W) \;\sim\; \frac{\det(N) \cdot Y \cdot F}{\det(D) \cdot E} \;\;. \tag{34}$$

Here $\det(N)$, $\det(D)$, F, and Y are all in what may be called a "standard form". They are all types of expressions met in [1]. Each is a gauge-invariant expression that may be ultraviolet cutoff by eliminating wavelets below some length scale and yielding a gauge-invariant cutoff regularization. (Each such truncation is gauge-invariant.) In these developments one always is working with matrices that are the identity if the gauge field is zero. One finds the gauge field contributions as traces of closed line integrals of the gauge field, with possible F field inserts, manifestly gauge-invariant expressions.

E is more complicated than the matrices treated in [1]. Any truncation of it is gauge-invariant, but it does not become the identity if the gauge field is zero. We let E_0^{TR} be the corresponding matrix with the gauge field set zero. We also set

$$V = E^{TR} - E_0^{TR}. \tag{35}$$

We then have

$$\det(E^{TR}) \;=\; \det(E_0^{TR} + V) \tag{36}$$

$$=\; \det(E_0^{TR}) \det\left(1 + (E_0^{TR})^{-1}\, V\right) \tag{37}$$

$$\sim\; \det\left(1 + (E_0^{TR})^{-1}\, V\right) \tag{38}$$

This is a good expression from which to compute contributions of E in perturbation theory. But, (38) is not as friendly an expression to get estimates from as the "standard

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forms" met in [1]. (There are alternate ways to treat E other than the development in (38).) **If** anomalies rear their ugly head, they will arise **only** from treachery concealed in E.

THE MATRIX $(E_0^{TR})^{-1}$

We abbreviate (E_0^{TR}) as Z. We see that

$$Z_{\alpha,\beta} = \left\langle \psi_\alpha , \ \frac{(-\Delta)^{1/2}}{W_0} \ \psi_\beta \right\rangle \tag{39}$$

We proceed to find a convenient expression for the inverse of Z. We define

$$Q_{\alpha,\beta} = \left\langle \psi_\alpha, \ \frac{W_0}{(-\Delta)^{1/2}} \ \psi_\beta \right\rangle . \tag{40}$$

$Q_{\alpha,\beta}$ is taken with the same truncation as $Z_{\alpha,\beta}$. (They are restricted to the same subset of wavelets.) If there were no truncation, Z and Q would be inverses (as well as conjugate transposes) of each other. It is natural to then write

$$Z^{-1} \ = \ (1+e)Q \tag{41}$$

where e is a "small" matrix, zero if no truncation. We then have

$$Z^{-1}Z \ = \ I \tag{42}$$

$$(1+e)QZ \ = \ I \tag{43}$$

$$e \ = \ (I - QZ)(QZ)^{-1}. \tag{44}$$

If we take a "sharp" ultraviolet cutoff, keeping all wavelets with length scales, ℓ, such that $\ell \geq \ell_0 = \frac{1}{2^{r_0}}$, and discarding wavelets with length scales $\ell < \ell_0$, we find the following properties of our matrices:

1) $(I - QZ)_{\alpha,\beta}$ is zero unless both α and β are at level ℓ_0. This fact depends on the property that Y. Meyer wavelets have of having no overlap in momentum space between wavelets differing by more than one level, and the diagonality of $(-\Delta)^{1/2}/W_0$ in momentum space.

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2) $(QZ)_{\alpha,\beta}$ is the identity for α, β at length scales $\ell > \ell_0$, has zero coupling between levels $\ell > \ell_0$ and level ℓ_0.

It follows that Z^{-1} couples wavelets with length scales differing at most by one level. To obtain good estimates we also want that

$$|Z^{-1}_{\alpha,\beta}| \le c_n \frac{\ell_\alpha^n}{|\gamma_\alpha - \gamma_\beta|^n} \tag{45}$$

for all $n > 0$ and some set of c_n. That is, we want matrix elements to fall off faster than any power of the distance between the centers of the wavelets, as measured in the length scale of the wavelets. This estimate is certainly true except possibly when either α or β are at the bottom level, ℓ_0 (since Q satisfies the estimate). To ensure this estimate we modify the truncation of E so that

1) we keep all wavelets with $\ell > \ell_0$

2) discard all wavelets with $\ell < \ell_0$

3) at level ℓ_0 we keep half the wavelets, the black squares of a checkerboard pattern.

We leave to a later publication showing that this ensures estimate (45). (It is not necessary for our other truncations to share this modification from a "sharp" cutoff.)

FINAL REMARKS

We restrict our observations to the perturbative regime.

$$\ell n \ (\det(W)) \ = \ T_1 + T_2 + \cdots \tag{46}$$

where T_n is a homogeneous polynomial of degree n in the A_μ field.

1) For $n \ge 5$ the gauge-invariant cutoff (regularized) T_n converge as the cutoff is removed. This is easy. We will want to prove that the limit for T_n agrees with any other calculation of these terms, that do not require renormalization.

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2) For $n \leq 4$ we have gauge-invariant cutoff (regularized) expressions for the T_n. We want to compute the limits, subtracting gauge-invariant counterterms if necessary, and prove the limits are Euclidean invariant. These may be difficult computations. **If** there are any difficulties with anomalies, it will be here.

3) We wish finally to emphasize a feature of our regularizations, following from properties of wavelets and the constructions employed. If one looks at two different cutoffs ℓ_0 and $\ell_0' < \ell_0$, then the cutoff expressions for T_n, $T_n(\ell_0)$ and $T_n(\ell_0')$, will differ by terms localized on a length scale ℓ_0. Connected diagrams for the difference have kernels rapidly going to zero when vertices separate measured on length scale ℓ_0 analogous to equation (45)). This was one goal of our constructions.

REFERENCES

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